An optimized trajectory for a UAV to track a helicopter

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1 Introduction

The United States Coast Guard has expressed an interest in using UAVs (unmanned aerial vehicles) to assist in helicopter-based search and rescue missions. In the application of interest, the helicopter flies in a square expanding spiral pattern in order to visually scan a search area. UAVs could assist in this operation by following the helicopter in its search pattern and scanning the water using heat-detecting cameras, thus increasing the area searched per second and allowing more area to be searched in less time.

The problem description does not allow modifications to the helicopter hardware or flight path. This means that the UAV control cannot depend on any direct knowledge of the helicopter state. Instead, a GPS receiver will be placed in the helicopter, along with a self-contained central controller for the UAVs (a laptop computer). Thus, UAV control will be based on Kalman filter estimation of the helicopter state based on GPS position measurements.

The UAV will be a small fixed-wing aircraft such as a SigRascal, which will be represented using the following kinematic model, with system inputs $\dot{\theta}$ (pitch rate) and $\dot{\psi}$ (yaw rate). x, y and z are the UAVs cartesian (north, east, up) coordinates.

$$
\begin{aligned}
\dot{x} &= V * cos(\theta) * cos(\psi) \\
\dot{y} &= V * cos(\theta) * sin(\psi) \\
\dot{z} &= V * sin(\theta) \\
\theta &\leq maximum\ climb\ slope \\
\theta &\geq maximum\ dive\ slope \\
abs(\dot{\psi}) &\leq maximum\ turn\ rate
\end{aligned} \tag{1}
$$

This model is reasonably accurate when describing a UAV being flown by an onboard autopilot system, such as the Piccolo, by Cloud Cap Technologies. The autopilot accepts a series of waypoints as inputs, and the UAV will fly from waypoint to waypoint with a constant forward velocity, if the trajectory satisfies the constraints above. For the current problem, the UAVs will fly at a constant altitude, and aircraft performance constants will come from a SigRascal aircraft with Piccolo autopilot, resulting in the following simplified model.

$$
\begin{aligned}
\dot{x} &= 20 * cos(\psi)\ \frac{m}{s} \\
\dot{y} &= 20 * sin(\psi)\ \frac{m}{s} \\
abs(\dot{\psi}) &\leq 10\ degrees\ per\ second
\end{aligned} \tag{2}
$$

The helicopter is represented by the standard double-integrator model, in which the three components of acceleration are the inputs.

$$\ddot{\vec{x}} = \vec{a} \tag{3}$$

However, for the purposes of this investigation, the helicopter will always be flying in the expanding spiral search pattern [2]. Each leg of the spiral is flown at a constant velocity. Each right-angle turn consists of a constant deceleration, an instantaneous right-angle turn, and a constant acceleration back to cruising velocity. Altitude remains constant. The spacing between legs (d) is determined by the altitude and the angle of the sensor sweep.

$$\begin{aligned} d &= 0.95 * A\, tan(\frac{\beta}{2}) \qquad (4) \\ A &= altitude \\ \beta &= sensor\ sweep\ angle \end{aligned}$$

If the UAVs are commanded to simply fly on either side of the helicopter at all times, they will attempt to follow a trajectory that does not satisfy their motion constraints. As a result, they will fall behind the helicopter, the search coverage will decrease, and collison with the helicopter may result, as shown below.

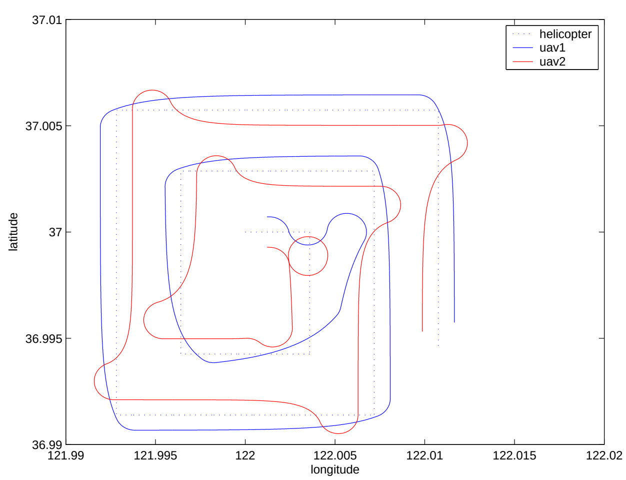

Figure 1: Simluation of two UAVs tracking a helicopter flying in expanding square pattern. UAV waypoints are fixed on either side of helicopter

A better alternative is to command the UAV to turn the corner along the optimum trajectory allowed by its motion constraints. The optimum trajectory is defined as the minimum-length feasible path from the UAV's position and orientation at the beginning of the turn to the UAV's desired position and orientation at the end of the turn. It has been shown that for this type of constraint, this optimum path consists of two minimum-turn-radius arcs connected by a straight line [1]. If, when not turning a corner, the UAV is commanded to fly a fixed distance ahead of the helicopter, the helicopter and UAV paths cross once instead of twice, decreasing the possibilities for collisions. The resulting UAV and helicopter trajectories for the right-angle turn maneuver are shown below. This maneuver is considered to begin when the helicopter begins decelerating for the turn, and ends when the UAV is again flying parallel to the helicopter after the turn.

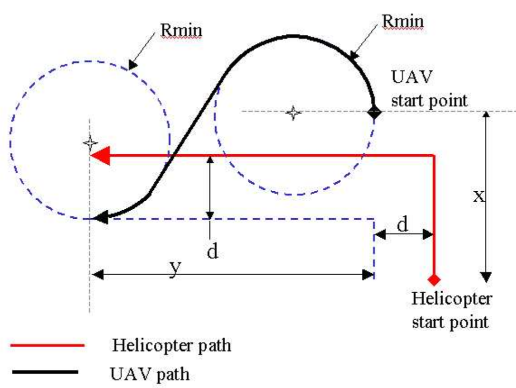

Figure 2: Proposed corner turn maneuver with helicopter and UAV trajectories

The distance x is the distance by which the UAV leads the helicopter when they are not in the process of turning a corner. The distance y represents where the UAV returns to its track parallel to the helicopter after turning a corner. These two distances will be chosen using optimization techniques.

2 Problem Formulation

One can see that the UAV path crosses in front of the helicopter before returning to its parallel path. The goal is to maximize the distance between the UAV and the helicopter when this occurs by choosing the free variables x and y. This goal function can be more simply formulated by maximizing the time difference between when the UAV reaches the path crossover point and when the helicopter does. Thus, the problem geometry leads to the following goal function.

$$\begin{aligned}
f(x) &= \frac{(\frac{3V^2}{2a}+d+R)(\Delta_2 R+\frac{1}{2}\Delta_1\sqrt{S})+(x-\frac{V^2}{2a}-R)(\Delta_1 R+\frac{1}{2}\Delta_2\sqrt{S}}{V(\Delta_2 R+\Delta_1\frac{1}{2}\sqrt{S})} \\
\Delta_1 &= x-\frac{V^2}{2a}+d-R \qquad (5) \\
\Delta_2 &= y-R \\
S &= \Delta_1^2+\Delta_2^2-4R^2 \\
a &= \textit{helicopter acceleration during turn} \\
d &= \textit{distance between helicopter and UAV track} \\
R &= \textit{UAV minimum turn radius} \\
V &= \textit{helicopter and UAV cruising speed}
\end{aligned}$$

The free variables x and y are governed by simple inequality constraints. The first must be a non-negative number, and the second must be sufficiently large so that the UAV can return to its parallel path by making minimum-radius turns. The third (equality) constraint presents more difficulty, but represents the fact that the UAV and the helicopter must complete their turn maneuvers in the same amount of time in order to return to their steady state formation after the turn. Therefore, the problem constraints are as follows.

$$x \geq 0 \qquad (6)$$

$$y-3R = g(y) \geq 0 \qquad (7)$$

$$\begin{aligned}
h(x,y) &= 0 \qquad (8) \\
&= \sqrt{S}-2R(cos^{-1}\left(\frac{2R}{\sqrt{\Delta_1^2+\Delta_2^2}}\right)+tan^{-1}\left(\frac{\Delta_1}{\Delta_2}\right))-y+x+c \\
c &= \frac{3\pi R}{2}-\frac{3V^2}{2a}-d
\end{aligned}$$

3 Grid Search

Due to the complication of the equality constraint and value function, an initial analysis was done by "brute force". The intial search space,

$$\{(x, y)|\ 0 \leq x \leq 100,\ 300 \leq y \leq 450\}$$

was filled by a unit search grid, and the constraint equation $h(x, y)$ was evaluated at each point on the grid. It can be seen that this search space satisfies the inequality constraints. It was then observed that the constraint was satisfied only in the neighborhood of the line

$$\{(x, y)|\ 0 \leq x \leq 20,\ y = -2.4 * x + 340\}$$

so a much finer search grid was created to fill a band along that region.

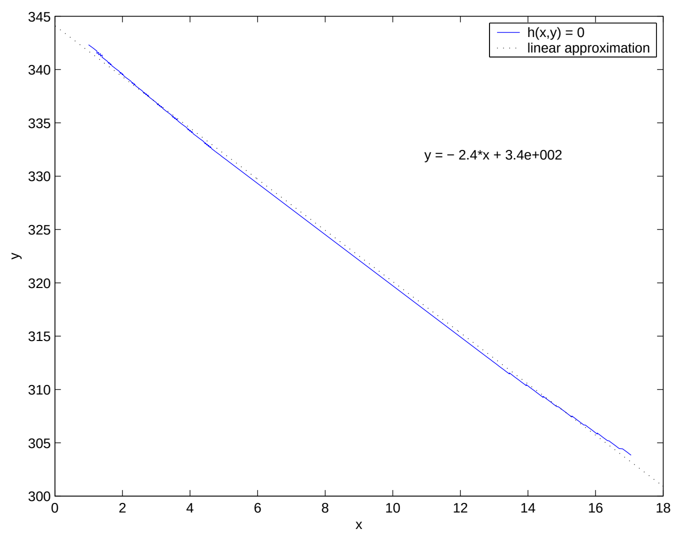

Figure 3: Feasible set and linear approximation

Again, $h(x, y)$ was evaluated at each point, and the feasible set

$$\{(x, y)|\ |h(x, y)| < 0.05\}$$

was selected. Then, $f(x, y)$ was evaluated at each point in the feasible set, and from these, the optimal point and value were selected.

$$\begin{aligned} (x^*, y^*) &= (17.05, 303.84) \\ f^*(x^*, y^*) &= 17.02 \\ h(x^*, y^*) &= 0.05 \end{aligned}$$

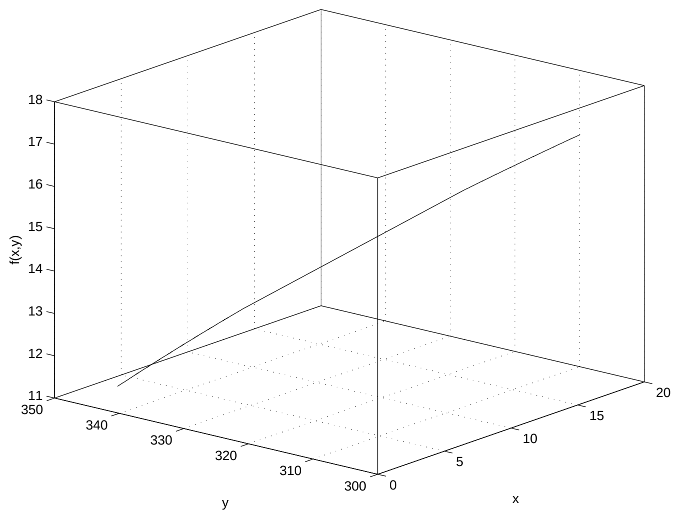

Figure 4: Value function f(x,y) on feasible set

4 Lagrangian Analysis

Lagrangian analysis can be used to lower bound the optimal value of a minimization problem. For the minimization of $f(x)$ subject to $h(x) = 0$ and $g(x) \leq 0$, the Lagrangian is written as follows

$$L(x, \lambda, \mu) = f(x) + \lambda h(x) + \mu g(x) \tag{9}$$

and the solution to the original optimization problem (the primal optimal, $p*$) is subject to the follwing.

$$\underset{x}{inf}\; L(x, \lambda, \mu) \leq p^* \tag{10}$$

For the current maximization problem,

$$\begin{aligned} p^* &= -min(-f(x)) \\ L(x, y, \lambda, \mu) &= -f(x) + \lambda h(x) - \mu_x x - \mu_y (y - 3R) \end{aligned}$$

$$\underset{x,y}{inf}\; \{-f(x, y) + \lambda h(x, y) - \mu_x x - \mu_y (y - 3R)\} \leq max(f(x, y)) \tag{11}$$

with the last equation holding true for any feasible choice of (λ, μ_x, μ_y). If one can find $(\lambda^*, \mu_x^*, \mu_y^*)$ such that

$$\underset{x,y}{inf}\; L(x, y, \lambda^*, \mu_x^*, \mu_y^*) \geq p_{des}^*$$

where p^*_{min} is the minimum allowable value of $f(x, y)$ for the UAV, then one can guarantee that (x^*, y^*) is an allowable solution to the UAV trajectory problem, even without having found the optimum point.

$$(x^*, y^*) = argmin\{L(x, y, \lambda^*, \mu_x^*, \mu_y^*) \tag{12}$$

It would be even more informative to be able to calculate p^* exactly (and therefore find the optimum UAV trajectory) rather than only to lower bound it. This could be accomplished using strong duality, but only in the cases where strong duality applies. Sufficient (but not necessary) conditions for strong duality to hold are the following two.

1. The problem is convex

2. Constraint qualification

For the problem to be convex, the value function and inequality constraints must be convex, and the equality constraint must be affine. In this case, the inequality constraints are affine, so they are convex, but the conditions on the value function and equality constraint are more difficult to analyze. However, since the state space is only two-dimensional, $f(x, y)$ and $h(x, y)$ can easily be plotted and examined visually. Upon inspection, $h(x, y)$ appears affine, which agrees with its linear approximation in the Grid Search section. Also upon inspection, $f(x, y)$ seems to be convex. This is of course not a rigorous analysis, but it is enough to suggest that further investigation in this direction could be worth-while.

Constraint qualification can be shown by Slater's Condition:

$$\exists x \in rel.\ int\ (dom(f))\ s.t.\ g_i(x, y) < 0, \forall\ g_i \tag{13}$$

This definition is for a problem with only inequality constraints. To extend it to an equality-constrained problem, it seems reasonable to replace the relative interior of dom(f) with the relative interior of $\{(x, y)|(x, y) \in dom(f), h(x, y) = 0\}$. In the case of this problem, this set occurs along the feasible line found by grid search, excluding its endpoints. Along this set, the inequality constraints will be strictly satisfied, so Slater's condition holds.

One can now postulate that strong duality holds, although it has not been proven. If strong duality does in fact hold, then the dual optimum is equal to the primal optimum, and the KKT conditions hold at the optimal point if the optimal point is regular. The KKT conditions can then be used to calculate the dual optimum, which is defined as follows.

$$q^* = \max_{\lambda, \mu_x, \mu_y} \min_{x, y} L(x, y, \lambda, \mu_x, \mu_y) \tag{14}$$

The KKT conditions come from the fact that the dual optimum must optimize $L(x, y, \lambda, \mu_x, \mu_y)$ and the primal problem, must satisfy the inequality and equality constraints, and $p^* = q^*$, leading to the following conditions.

$$\begin{aligned} g_j(x, y) &\leq 0 \\ h(x, y) &= 0 \\ \mu_x \; and \; \mu_y &\geq 0 \\ \mu_j g_j &= 0 \\ \nabla f(x^*, y^*) + \sum \mu_j \nabla g_j(x^*, y^*) + \lambda \nabla h_i(x^*, y^*) &= 0 \end{aligned} \tag{15}$$

The KKT conditions can be used to search for a solution to the dual optimum problem by representing it as the zero-finding problem $G(z) = 0$ where

$$G(z) = \begin{bmatrix} \nabla f(x, y) + \lambda \nabla h(x, y) - \mu_x - \mu_y \\ h(x, y) \end{bmatrix} \tag{16}$$

$$z = \begin{bmatrix} x \\ y \\ \lambda \\ \mu_x \\ \mu_y \end{bmatrix}$$

However, in order to assure that the KKT conditions hold and there exist unique Lagrange multipliers, the optimal point must be regular. In the case of this problem, there are three constraints and only two unknowns, so regularity requires that at least one inequality constraint be inactive. This can be assumed, because we want $x > 0$. Now the regularity conditions becomes linear independence of $\nabla h(x^*, y^*)$ and $\nabla(3R - y)$:

$$\nabla h(x, y) \notin span\{\begin{bmatrix} 0 \\ -1 \end{bmatrix}\} \tag{17}$$

Unfortunately, at the optimum point found in the Grid Search section, $\nabla h(x^*, y^*) = [0 \; 1]^T$, so there may not exist unique Lagrange multipliers that satisfy the KKT conditions.

Alternatively, one can solve for the dual optimum directly as a simultaneous minimization and maximization problem, as defined in equation (14). Because the Lagrangian must be minimized over the primal variables and maximized over the dual variables, a gradient-based method such as Newton's method could be applied as follows.

1. Start at initial condition (x_o, λ_o, μ_o)

2. Apply Newton's method to

$$(\lambda^*, \mu^*) = arg\, max\{L(x_o, \lambda, \mu)\}$$

3. Apply Newton's method to

$$x^* = arg\, min\{L(x, \lambda^*, \mu^*)\}$$

4. Continue alternating minimization and maximization using the results of the previous iteration

This method does not depend on the KKT conditions or regularity of the optimal point, although it still depends on strong duality. If strong duality is not the case, this method will not converge due to the non-zero duality gap.

5 Conclusions

Although the maximization problem has only two variables and three constraints, it is difficult due to the complexity of the value function and the constraint function $h(x, y)$. Many of the standard optimization techniques, such as conic programming, quadratic programming, and linear programming, cannot be applied. By searching the state space for points that satisfy the constraints, it was observed that the feasible set is approximately affine and the value function is approximately convex. This suggests that a min-max gradient-based search algorithm such as described above could converge to the global optimum without getting stuck at local optima. However, lacking the time to implement this, one can be reasonably confident that the grid-based search conducted initially gives a good approximation of the optimal solution.

References

[1] L.E. Dubbins. On Curves of minimal length with a constraint on average curvature and with prescribed initial and terminal poisitions and tangent. *American Journal of Mathematics*. 79:497-516. 1976

[2] *U.S. Coast Guard Addendum to the United States National SAR Supplement* (CGADD), COMDTINST M16130.2C, Chapter 3 and Appendix H, available at: http://www.uscg.mil/hq/g-o/g-opr/manuals.htm